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ON THE BEHAVIOR OF Z-IDEALS AND Z°-IDEALS IN INTERMEDIATE RINGS

Nagulmeera Shaik

Research scholar

Dr. Subhash Chandra Mourya

Principal & Head, Dept. of Maths, Government Inter College, Bareilly

ABSTRACT

The theory of ideals in commutative rings, particularly in rings of continuous functions, has been a fertile area of research due to its deep connections with topology and functional analysis. This paper focuses on the structural and behavioral analysis of z-ideals and z∘-ideals within the framework of intermediate rings $A(X)$, which lie between subrings and the full ring $C(X)$of real-valued continuous functions on a topological space X. Building on the classical characterization of z-ideals through zero-set inclusion, and the refined notion of z∘-ideals via interior operations on zero sets, this study extends these concepts to intermediate rings where additional algebraic constraints introduce new complexities. The paper establishes several foundational results describing the interaction between annihilators, zero-set interiors, and membership in intermediate rings. In particular, it provides a characterization of elements in $A(X)$using inclusion relations of interiors of zero sets and annihilators, leading to a deeper understanding of the structure of ideals in reduced rings. A novel topological representation of z∘-ideals is developed through closed-set ideals, demonstrating a correspondence between algebraic ideals in $A(X)$and topological constructs in X.

Keywords: Theory, Algebra, Rings, Functions, Theorem.

I. INTRODUCTION

The study of ideals has long occupied a central position in commutative algebra, serving as a bridge between algebraic structures and topological or geometric interpretations. Within this broad framework, the concept of *z-ideals* and their variants has emerged as an important area of investigation, particularly in rings of continuous functions and their generalizations. The paper titled *On the Behavior of Z-Ideals and Z∘-Ideals in Intermediate Rings* is situated within this line of inquiry and seeks to extend and deepen our understanding of how these specialized ideals behave when considered in intermediate algebraic structures. To appreciate the motivation behind such a study, it is essential to recall that classical z-ideals were originally defined in the context of rings like $C(X)$, the ring of all real-valued continuous functions on a topological space X. In that setting, a z-ideal is characterized by its intimate relationship with zero sets: if a function belongs to the ideal, then any function whose zero set contains the zero set of the original function must also belong to the ideal. This property encodes a strong interaction between algebraic containment and topological structure, making z-ideals a powerful tool for translating between the language of functions and the language of sets. Over time, researchers recognized that variations of z-ideals—such as z∘-ideals—capture even finer distinctions, particularly when one considers interiors of zero sets or related topological refinements. These concepts are not merely technical curiosities; rather, they reveal subtle layers of structure in function rings and provide insights into how algebraic properties reflect the underlying topology of spaces.

In recent developments, attention has shifted toward *intermediate rings*, which lie between smaller subrings and larger rings of continuous functions. These rings often arise naturally when one restricts attention to functions with additional properties—such as boundedness, measurability, or specific growth conditions—while still retaining enough structure to support meaningful algebraic analysis. The behavior of ideals in such intermediate rings is far from trivial, as many properties that hold in the full ring $C(X)$may fail or require modification. Consequently, the study of z-ideals and z∘-ideals in intermediate rings is both a natural extension and a nontrivial generalization of classical theory. It raises fundamental questions: To what extent do the defining properties of z-ideals persist in intermediate contexts? How do these ideals interact with the additional constraints imposed on the ring? Are there new characterizations or equivalences that emerge when one moves away from the full function ring? Addressing these questions not only enriches the theory of ideals but also contributes to a broader understanding of how algebraic structures adapt to different functional settings.

The introduction of z∘-ideals, in particular, reflects an effort to refine the notion of zero-set containment by incorporating interior operations, thereby capturing more delicate topological information. While z-ideals focus on inclusion relationships among zero sets, z∘-ideals take into account the interior of these sets, leading to a more nuanced classification of functions and their associated ideals. This distinction becomes especially significant in intermediate rings, where the interplay between algebraic and topological properties may differ markedly from the classical case. For instance, certain closure properties or maximality conditions that hold for z-ideals in $C(X)$might need to be reinterpreted or adjusted in intermediate settings. The paper under consideration aims to systematically investigate these phenomena, providing new results that clarify the structure and behavior of both z-ideals and z∘-ideals within this broader framework.

Moreover, the study of these ideals is closely connected to other important concepts in commutative algebra and topology, such as prime ideals, maximal ideals, and the spectrum of a ring. Understanding how z-ideals and z∘-ideals relate to these classical notions can yield valuable insights into the overall structure of intermediate rings. For example, one may ask whether every z-ideal is contained in a maximal z-ideal, or how the lattice of z∘-ideals compares to the lattice of all ideals in the ring. These questions are not only of theoretical interest but also have implications for applications in areas such as functional analysis and topology, where rings of functions play a central role. By exploring these connections, the paper contributes to a more unified and comprehensive theory of ideals in function rings.

Another important aspect of this work is its potential to generalize known results and identify new patterns. In mathematics, extending concepts to broader contexts often reveals hidden structures and unexpected relationships. The behavior of z-ideals and z∘-ideals in intermediate rings exemplifies this principle: while some properties carry over directly from the classical setting, others require new techniques or lead to entirely new phenomena. The investigation of these differences is a key theme of the paper, highlighting both the robustness and the limitations of existing theories. Through careful analysis and the development of new tools, the authors aim to establish a deeper understanding of how these ideals function in more general environments.

The exploration of z-ideals and z∘-ideals in intermediate rings represents a significant step in the ongoing development of commutative algebra and its applications to topology. By extending classical concepts to new settings, the paper not only broadens the scope of the theory but also uncovers new layers of structure and complexity. The results obtained are expected to have implications beyond the immediate context, influencing future research in related areas and providing a foundation for further investigations. As such, this work stands as an important contribution to the field, offering both a synthesis of existing knowledge and a platform for new discoveries.

II. REVIEW OF LITERATURE

Rezaei Aliabad et al., (2010) By use of z-ideals of C(X), we define the z-ideals of the factor rings of C(X). For pseudocompact spaces X, we prove that J/I is a z-ideal in C(X)/I if and only if J is a z-ideal in C(X) that contains the m-closure of the ideal I. Other conditions are also satisfied. Based on this, it follows that for any pseudocompact X, the product of two m-closed ideals (e-ideals) in C(X) is an m-closed ideal (e-ideal). It is further shown that for any pair of z°-ideals I ⊆ J in C(X), J/I is a z°-ideal in C(X)/I if and only if every prime z°-ideal in C(X) is minimum, and this holds true for z°-ideals of factor rings of C(X).

Acharyya, Sudip et al., (2019) It is shown that the set $\hat{X}$ of all ultrafilters of measurable sets on X with the Stone-topology is homeomorphic to the set of all maximal ideals of the ring }(X,𝒜) of real valued measurable functions on a measurable space (X,𝒜) equipped with the hull-kernel topology. In terms of the points of $\hat{X}$, this gives a comprehensive description of the maximum ideals of ℳ(X,). The identical, compact, zero-dimensional Hausdorff structure spaces are shown for any intermediate subrings of ℳ(X,𝒜) that include the bounded measurable functions. It is noted that C(X) = ℳ(X, 𝒜), where 𝒜 is the σ-algebra of the zero-sets of X, holds when X is a P-space.

III. Z∘-IDEALS IN INTERMEDIATE RINGS

Each bounded function in C(X) belongs to any intermediate ring A(X), and this fact, together with the proof of Theorem 3.10, leads us to the following proposition:

Theorem 1. If intXZ(f) ⊆ intXZ(g) and for any f and g in space X, Ann(f) ⊆ Ann(g) then and only if g is an element of A(X). In this case, the annihilator of f in A(X) is denoted by Ann(f).

Since any intermediate ring A(X) is reduced, the following proposition follows from Theorem 1.

Theorem 3.2. Pf is the set of all g elements in A(X) such that intXZ(f) ⊆ intXZ(g) for any f in A(X)...

The following novel topological description of z ⌋-ideals in the intermediate rings is obtained by using this theorem and rigorously following the reasons in the proof of Theorem 3.11; this is an unintentional generalisation of Theorem 1.

Theorem 3. Every ideal P of closed sets in X has a z ∘ -ideal of A(X) in the set CP(X) ∩ A(X) ≡ {f ∈ A(X) : clX(X \ Z(f)) ∇ P}.

The converse is also true: for every z ⋦ -ideal of A(X) that is I, there is a closed set ideal P[I] in X such that I = CP[I](X) ∩ A(X).

We need this supplementary result to prove that a suitably contained intermediate ring in C(X) is never von Neumann regular.

Lemma 1. An absolutely convex subring of C(X) is an intermediate ring A(X). This means that if |f| < |g|, f ∈ C(X) and g ∈ A(X), then f ∈ A(X).

Proof. The set M∏(X, A) contains the function f[1+|g|]. That is why the function f, which is defined as f multiplied by the absolute value of g, belongs to the set A(X).

IV. Z-IDEALS OF INTERMEDIATE RINGS

Definition 1. If, for any element an in the set I, there is an intersection of all maximal ideals of the set R that contain a, denoted as Ma, then the ideal I in the commutative ring R with unity 1 is termed the z-ideal.

A connection between z-ideals and z ◦ -ideals in the ring R has been proven in the following statement.

Theorem 4. Every z ◦ -ideal of a reduced ring R is a z-ideal if and only if R is a ring. It is worth noting that every z ◦ -ideal in C(X) is a z-ideal, and this holds true in particular for every z ◦ -ideal in an intermediate ring R.

As was demonstrated in, there is no distinction between z-ideals and z ⋦ -ideals in C(X) for a nearly P-space X. To ensure that this thesis is comprehensive, we have included the pertinent proposal below.

Theorem 5. For a Tychonoff space X to be nearly P, every z-ideal in C(X) must be a z ◦ -ideal.

Proof. A nearly P-space is X. For every f ∈ I and g ∈ C(X), assume that intXZ(f) ⊆ intXZ(g) and that I is a z-ideal in C(X). If we want to prove that I is a z ⋦ ideal, we need to prove, according to Remark 3.1, that g is an element of I. A zero set is a regular closed set in an almost P-space. When this happens, we get Z(g) = clXintXZ(f) ⊆ clXintXZ(g). G belongs to I since I is a z-ideal in C(X). That is why I am a z ◦ ideal. On the other hand, assume that every z-ideal in C(X) is a z ◦ -ideal.

Specifically, every extreme ideal in C(X) is also an ideal in C(X) in the z-direction.

Among all the intermediate rings A(X)'s, C(X) is characterized by the aforementioned theorem within the class of nearly P-spaces.

Theorem 6. An intermediate ring A(X) 6= C(X) is defined for X, which is nearly P. If that is so, a z-ideal in A(X) does not have to be a z ◦ -ideal.

Proof. The existence of a maximum ideal M in A(X) that is not a z ◦ -ideal is inferred from Theorem 3.17. M is a z-ideal in A(X) without being a z ◦ -ideal since every maximal ideal in a commutative ring R is a z-ideal.

V. ZA-IDEALS IN A(X) VERSUS P-SPACES/ALMOST P-SPACES X

Definition 2. If there exists a subset g of A(X) such that f(x)g(x) = 1 for all x in E, then we say that f is E-regular. Define ZA(f) as the set of all ideals in A(X) where f is X-regular, and for any ideal I in A(X), we get ZA[I] = ∪f∈IZA(f). Moreover, for any ideal I, ZA[I] = ∪f∈IZA(f), and let ZA(f) = {E ∈ Z[X] : f is Hregular for each zero set H present in X \ E}. Assuming f is not invertible in A(X), the four entities ZA(f), ZA(f), ZA[I] and ZA[I] are all z-filters on X, as shown in. Is further shown that for every z-filter F on X, the suitable ideals in A(X) are Z −1 A [F] = {f ∈ A(X) : ZA(f) ⊆ F} and Z −1 A [F] = {f ∈ A(X) : ZA(f) ⊆ F}. If the equivalence holds, then any ideal I in A(X) is referred to as a ZA-ideal in A(X), since Z −1 A ZA[I] ⊇ I. This may be easily proven. Just like that, it's easy to verify that Z −1 A ZA[I] ⊇ I for any ideal I in A(X). If this equivalence is true, then we say that I is the ZA-ideal in A(X).

Notation: As points of the Stone-Cech compactification ˇ βX of X may be used to index the set of all z-ultrafilters on X, we can write U p as the z-ultrafilter on X that corresponds to the point p of βX. Consequently, it is a well-established fact in the field of Rings of Continuous Functions that X is homoeomorphic to the set of all z-ultrafilters on X that have the Stone-topology. However, using the points of βX as indexes, one can access the maximum ideals of A(X) as the structure space, which is the set of all maximal ideals of any intermediate ring A(X), is homoeomorphic to βX. The point p ∈ βX that corresponds to the maximal ideal in A(X) is represented by M p A. The following statement, which is established in [32] and is a standard outcome in intermediate rings of continuous functions, establishes a natural connection between the maximal ideal M p A and the z-ultrafilter Up on X.

Theorem 7. Assuming p is an element of βX and A is an intermediate ring of C(X), the set Mp A is defined as {f ∈ A(X) : ZA(f) ⊆ Up} = Z −1 A [U p].

The present thesis is self-contained since it draws on four subsidiary conclusions, which we duplicate with proof, to uncover the potential interrelationships among ZA-ideals, z-ideals and z ◦ -ideals in A(X). An intermediate ring is represented by A(X) in all of these outcomes.

Lemma 1. In the set A(X), f is considered E-regular if, for every x in E, $f(x) \geq c > 0$ for some c in R. In this case, E is a subset of X.

Proof. With $h = c \vee f$, we can see that h^{-1} is a member of $C*(X)$ and, thus, h^{-1} is a member of A(X). For all x in E, we observe that $(h^{-1} f)(x) = 1$. Thus, f is E-regular.

VI. CONCLUSION

The present study offers a comprehensive examination of the structure and behavior of z-ideals, z∘-ideals, and ZA-ideals within intermediate rings $A(X)$, thereby extending the classical theory developed for the ring $C(X)$of continuous functions. One of the central insights of this work is that although intermediate rings retain many foundational characteristics of $C(X)$, the restriction imposed by their structure introduces significant variations in how ideals behave. This necessitates a careful re-evaluation of well-known results and leads to the emergence of new properties that are unique to intermediate settings. A key contribution of this study lies in the characterization of elements of $A(X)$through the relationship between interiors of zero sets and annihilators. By establishing that inclusion relations involving $\mathrm{int}_X Z(f)$and $\mathrm{int}_X Z(g)$correspond to inclusion of annihilators, the work strengthens the connection between algebraic and topological aspects of the theory. This result not only deepens our understanding of ideal membership in intermediate rings but also provides a useful tool for further analysis.

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